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# Technical Report

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## Optimum Detection of M-ary Orthogonal Signals in ELF Noise Environments

J. W. Modestino

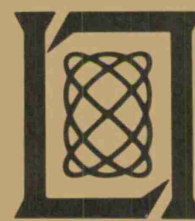
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OPTIMUM DETECTION OF M-ARY ORTHOGONAL SIGNALS  
IN ELF NOISE ENVIRONMENTS

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#### ABSTRACT

Explicit receiver structures for the optimum detection of M-ary orthogonal signals in impulsive noise environments typical of ELF are determined. The resulting optimum structures, while shown to bear some resemblance to that which would have been obtained in the presence of Gaussian noise alone, exhibit an interesting nonlinear behavior. The difficulties in the actual implementation of this receiver are discussed, and several definitely suboptimum yet computationally superior structures are suggested. The results are felt to be applicable to a much broader class of channels characterized by impulsive interference.

Accepted for the Air Force  
Joseph R. Waterman, Lt. Col., USAF  
Chief, Lincoln Laboratory Project Office



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# OPTIMUM DETECTION OF M-ARY ORTHOGONAL SIGNALS IN ELF NOISE ENVIRONMENTS

## I. INTRODUCTION

In previous work,<sup>1</sup> a model for ELF noise was presented which consisted of the sum of a low-density shot process and white Gaussian noise (WGN). In the present report, we consider the structure of optimum receivers for the detection of M-ary orthogonal signals in background noise as described by this model. While it is not the intent here to suggest that the optimum detector structure should actually be employed in practice (generally it is quite complicated), instead the reasons for studying optimum processing structures are twofold:

- (a) To provide a benchmark against which the performance of suboptimum schemes can be compared,
- (b) To provide insight which may suggest computationally superior yet reasonably efficient suboptimum structures.

The binary detection problem has been considered previously.<sup>2</sup> We shall treat a more general situation where the channel signaling alphabet consists of M orthogonal equi-energy waveforms. The approach to be used follows closely the optimal dynamical filtering results associated with the names of Kalman and Bucy,<sup>3</sup> Kushner,<sup>4,5</sup> and Wonham.<sup>6</sup>

## II. PRELIMINARIES

The received signal is assumed of the form

$$r(t) = s(t) + n(t) \quad , \quad (1)$$

where  $s(t)$  is one of the a priori known signal waveforms  $s_i(t)$ ,  $i = 1, 2, \dots, M$  satisfying

$$\int_0^T s_i(t) s_j(t) dt = E_s \delta_{ij} \quad (2)$$

with  $\delta_{ij}$  the Kronecker delta, and T the observation interval. The additive signal independent noise  $n(t)$  appearing in Eq. (1) is a sample function of a non-Gaussian noise process of the form

$$n(t) = y(t) + w(t) \quad , \quad (3)$$

where  $w(t)$  is a zero-mean WGN process with variance  $\sigma_w^2 = N_o/2$  and  $N_o$  the double-sided noise spectral density in watts/Hz. The process  $y(t)$ , on the other hand, is a low-density\* shot process obtained by exciting a finite-state linear dynamical system by a random impulse train. In particular,  $y(t)$  is described by the state equation

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{b} u(t) \quad (4)$$

and the output relation

$$y(t) = \langle \underline{c}, \underline{x}(t) \rangle \quad , \quad (5)$$

where  $\underline{A}$  is a constant  $n \times n$  matrix,  $\underline{b}$  and  $\underline{c}$  are  $n$ -vectors (specifically column vectors) and  $\langle \cdot, \cdot \rangle$  represents an inner product. The state vector  $\underline{x}(t)$  is itself an  $n$ -vector while the quantity

\* By "low-density" we mean that the average inter-arrival times between exciting impulses is greater than any system time constants.

$u(t)$  appearing in (4) is a scalar impulse process independent of  $w(t)$  and of the form

$$u(t) = \sum_{i=1}^{N(t)} u_i \delta(t - t_i) \quad (6)$$

Here  $\{N(t), t \geq 0\}$  is a counting process<sup>7</sup> with arrival times  $t_1, t_2, \dots$  assumed Poisson distributed with intensity  $\lambda$  events/second, and  $\{u_i\}$  is a sequence of zero-mean independent and identically distributed (i.i.d) random pulse amplitudes with common univariate probability density function (p.d.f.)  $f(\cdot)$ . In Ref. 1, two different amplitude distributions were considered: the double-sided power-Rayleigh distribution with p.d.f.

$$f(u) = \frac{\alpha |u|^{\alpha-1}}{2R_0^\alpha} \exp \left\{ -\left| \frac{u}{R_0} \right|^\alpha \right\} \quad ; \quad 0 < \alpha \leq 2 \quad (7)$$

and the double-sided lognormal distribution with p.d.f.

$$f(u) = \frac{1}{2\sqrt{2\pi} \sigma |u|} \exp \left\{ -\frac{\ln^2 |u|}{2\sigma^2} \right\} \quad (8)$$

with  $\sigma^2$  the variance of a zero-mean Gaussian random variable  $g$  for which  $u = e^g$ .

The suitability of this model for representing ELF noise environments is discussed in some detail in Ref. 1 and need not be repeated. The significant parameters which must be specified then are:

- (a) The p.d.f.  $f(\cdot)$  of the pulse amplitudes,
- (b) The intensity  $\lambda$  of the Poisson process,
- (c) The linear system dynamics, i.e., the quantities  $\underline{A}$ ,  $\underline{b}$ , and  $\underline{c}$  in Eqs. (3) and (4),
- (d) The ratio  $\gamma$  of the rms value of the shot component to the Gaussian background component.

It is significant that these parameters can be adjusted quite easily to match observed ELF noise recorded at various geographical locations. Observe that the choice of background noise level  $\sigma_0^2$  is somewhat arbitrary and amounts to a trivial rescaling of the  $n(t)$  process.

### III. OPTIMUM RECEIVER STRUCTURE

It will be assumed that the receiver bases its decisions upon the relative likelihood that a particular signal has been sent and resulted in the observation  $r_{0,T}$  where, in general,  $r_{0,t} \triangleq \{r(\tau), 0 \leq \tau \leq t\}$ . In particular, the decision strategy is given by: announce signal  $i$  as having been transmitted  $i = 1, 2, \dots, M$  iff

$$p[r_{0,T}|i] = \max_{1 \leq j \leq M} p[r_{0,T}|j] \quad (9)$$

where  $p[r_{0,T}|j]$  is the probability (likelihood) of having received  $r_{0,T}$  given that the  $j^{\text{th}}$  signal  $j = 1, 2, \dots, M$  had been transmitted. It will prove convenient to normalize the likelihoods and in fact to base decisions upon the logarithm\* of the normalized likelihoods so that the modified decision rule becomes: announce signal  $i$  as having been transmitted  $i = 1, 2, \dots, M$  iff

\* Actually any monotonic function suffices.

$$\lambda_i[r_{0,T}; T] = \max_{1 \leq j \leq M} \lambda_j[r_{0,T}; T] \quad , \quad (10)$$

where\*

$$\lambda_j[r_{0,t}; t] \triangleq \ln \frac{p[r_{0,t}|j]}{p_0[r_{0,t}]} \quad ; \quad j = 1, 2, \dots, M \quad (11)$$

and  $p_0[\cdot]$  is the probability distribution of the WGN process  $w(t)$ . Note the explicit dependence of the log-likelihoods  $\lambda_j[\cdot; t]$  on the time  $t$ . Using some recent results of Kailath,<sup>8</sup> the log-likelihoods assume a particularly convenient form. More specifically, under the assumption

$$\int_0^T E\{y^2(t)\} dt < \infty \quad , \quad (12)$$

where  $E\{\cdot\}$  represents the expectation operator, Kailath demonstrates that

$$\lambda_j[r_{0,t}; t] = \int_0^t \hat{h}_j(\tau) d\eta(\tau) - \frac{1}{2} \int_0^t \hat{h}_j^2(\tau) d\tau \quad ; \quad j = 1, 2, \dots, M \quad , \quad (13)$$

where  $d\eta(t) = r(t) dt$ , and  $\hat{h}_j(t)$  represents the least mean-square estimate of the signal plus shot noise process  $h_j(t) = s_j(t) + y(t)$ . In particular, due to the additivity of the expectation operator,

$$\hat{h}_j(t) = s_j(t) + \langle \underline{x}, \hat{\underline{x}}(t|j) \rangle \quad , \quad (14)$$

where

$$\hat{\underline{x}}(t|j) = E\{\underline{x}(t) | r_{0,t}, j\} \quad ; \quad j = 1, 2, \dots, M \quad (15)$$

is the least mean-square estimate of the state vector  $\underline{x}(t)$  given that signal  $j$  has been transmitted and has resulted in the observation  $r_{0,t} = \{r(\tau), 0 \leq \tau \leq t\}$ . The first integral in (13) must be interpreted with care as a stochastic integral – an Ito integral to be precise. Nevertheless, the practical interpretation of this integral is as a cross-correlation operation between the received process and the least mean-square estimate of the signal plus shot noise component. The resulting receiver structure is illustrated in Fig. 1 which would be identical to the optimum detector for known signals in Gaussian noise were it not for the presence of the estimators required to recover the  $\{\hat{h}_j(t)\}$  from the received data. Indeed, from (14) if there were no shot noise present,  $\hat{h}_j(t) = s_j(t)$  independent of the received data and the detector reduces to the familiar correlator or matched filter receiver. The presence of the shot noise component, on the other hand, requires explicit computation of the estimates  $\{\hat{h}_j(t)\}$ , or equivalently,  $\{\hat{\underline{x}}(t|j)\}$ , and complicates the receiver implementation considerably. The remainder of this report will be concerned with the development of computational algorithms for generating the least mean-square estimates  $\{\hat{\underline{x}}(t|j)\}$  from the noisy received data.

Let us first recall from the definition of  $\hat{\underline{x}}(t|j)$  the explicit assumption that the  $j^{\text{th}}$  signal was transmitted and resulted in the observation  $r_{t_0,t}^\dagger$ , i.e., from (15)

$$\hat{\underline{x}}(t|j) = \int_{R^n} \underline{x} p(\underline{x}; t | r_{t_0,t}, j) d\underline{x} \quad ; \quad j = 1, 2, \dots, M \quad , \quad (16)$$

\* The fact that we are using the variable  $\lambda$  to designate the intensity of the Poisson point process and  $\{\lambda_j\}$  to designate log-likelihoods should cause no confusion.

† Note that we have explicitly assumed  $t_0 = 0$  above.

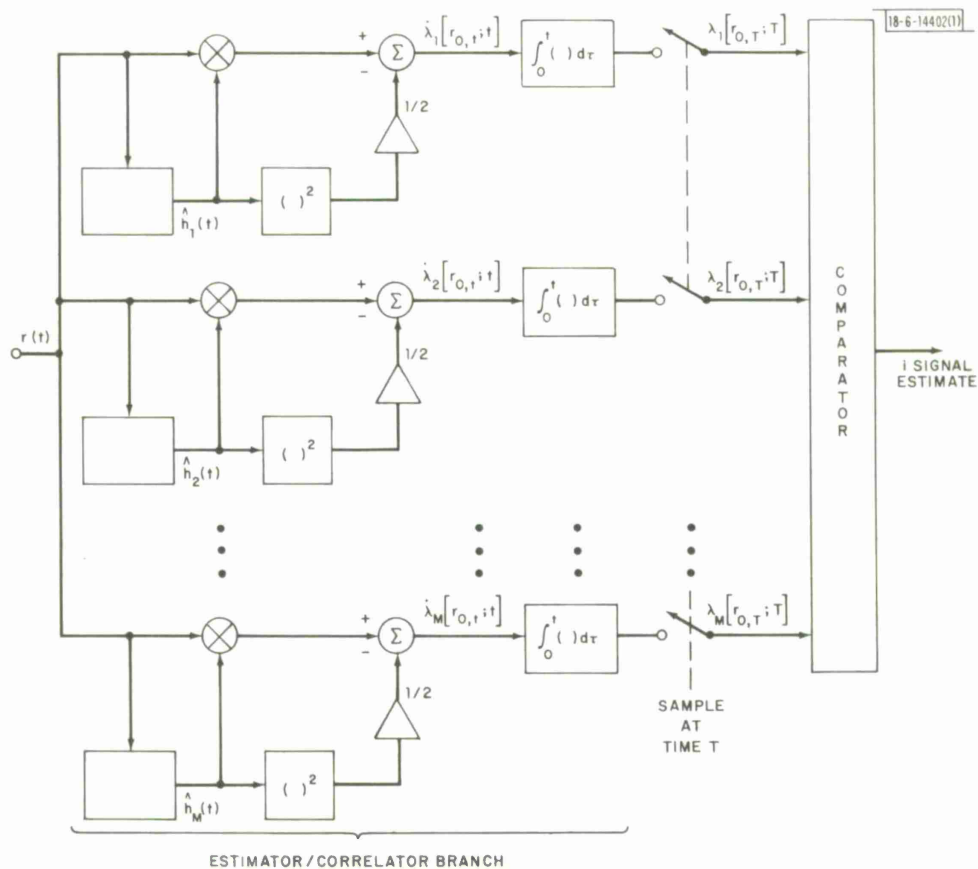


Fig. 1. Optimum detector structure.

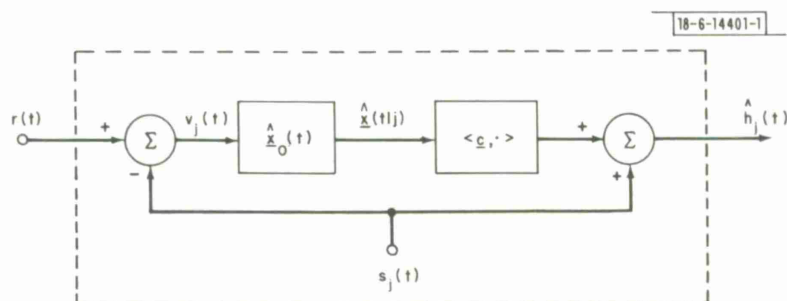


Fig. 2. Generation of least-squares estimates.

where the conditional probability density function  $p\{\underline{x}; t | r_{t_0, t}, j\}$  represents the probability of observing the state  $\underline{x}(t) = \underline{x}$  at time  $t$  given that the  $j^{\text{th}}$  signal was transmitted and has resulted in the observation  $r_{t_0, t}$ . The integration in (16) is over the  $n$ -dimensional Euclidean space  $R^n$  with volume element  $d\underline{x}$ . Note that we again explicitly include the time  $t$  in the conditional probability expression to emphasize its dependence upon time. It follows that

$$p\{\underline{x}; t | r_{t_0, t}, j\} = p\{\underline{x}; t | v_j(\tau), t_0 \leq \tau \leq t\} \quad ; \quad j = 1, 2, \dots, M \quad , \quad (17)$$

where  $v_j(t) = r(t) - s_j(t)$  since, given that the  $j^{\text{th}}$  signal was transmitted and resulted in the observation  $r_{t_0, t}$ , it is only this difference representing the additive noise process  $n(t)$  which is relevant to the determination or estimation of the state vector  $\underline{x}(t)$ . Thus, the estimators  $\hat{\underline{x}}(t | j)$ , or equivalently  $\hat{h}_j(t)$ ,  $j = 1, 2, \dots, M$  can be implemented as illustrated in Fig. 2. Here the quantity  $\hat{\underline{x}}_0(t)$  represents the least mean-square estimator of the state vector  $\underline{x}(t)$  given observations on the additive noise process  $n(t)$  in the interval  $[t_0, t]$ , i.e.,

$$\hat{\underline{x}}_0(t) = E\{\underline{x}(t) | n_{t_0, t}\} \quad . \quad (18)$$

Observe that the input to the estimator  $\hat{\underline{x}}_0(t)$  in Fig. 2 is actually  $v_j(t)$  which will be equal to  $n(t)$  only if the  $j^{\text{th}}$  signal was in fact transmitted. If, instead, the  $i^{\text{th}}$  signal was actually transmitted, then  $v_j(t)$  contains in addition to the noise process  $n(t)$  a bias term equal to the difference between the  $i^{\text{th}}$  and  $j^{\text{th}}$  signals. It will be convenient then to let  $v(t) = n(t)$  represent a generic input to the estimator in Fig. 2 which will equal  $v_j(t)$  in the  $j^{\text{th}}$  estimator branch. Similarly, we will drop subscripts in describing the estimator  $\hat{\underline{x}}_0(t)$  given by (18) and proceed then to evaluate

$$\hat{\underline{x}}(t) = \int_{R^n} \underline{x} p\{\underline{x}; t | v_{t_0, t}\} d\underline{x} \quad (19)$$

or more precisely the conditional probability density  $p\{\underline{x}; t | v_{t_0, t}\}$ . In the following section we will derive a modified Fokker-Planck equation which governs the time evolution of this conditional density. In particular, it will be shown that

$$\begin{aligned} \frac{\partial p\{\underline{x}; t | v_{t_0, t}\}}{\partial t} &= L^+ p\{\underline{x}; t | v_{t_0, t}\} + \sigma_0^{-2} p\{\underline{x}; t | v_{t_0, t}\} \\ &\quad \times [v(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle] [\langle \underline{c}, \underline{x}(t) \rangle - \langle \underline{c}, \hat{\underline{x}}(t) \rangle] \quad , \end{aligned} \quad (20)$$

where  $L^+\{\cdot\}$  is the forward differential generator of the shot process given in this case by

$$L^+\{\cdot\} = -\text{Tr} \left( \frac{\partial \underline{A} \underline{x}(t)}{\partial \underline{x}} \{\cdot\} \right) - \lambda(t) E [\Delta_u \{\cdot\}] \quad , \quad (21)$$

where  $\text{Tr}$  represents the trace of a matrix,  $\underline{A}$  is the  $n \times n$  matrix described in (4) and the sequel,  $\lambda(t)$  is the intensity\* of the exciting Poisson point process. The expectation in (21) is with respect to the pulse amplitude distribution with generic random variable  $u$ , and finally  $\Delta_u \{\cdot\}$  is the difference operator defined for any continuous functional on  $R^n$  by

$$\Delta_u \{g(\underline{x})\} = g(\underline{x}) - g(\underline{x} - \underline{b}u) \quad , \quad (22)$$

\*Note that we are allowing the intensity to be possibly a function of time.

where  $\underline{b}$  has been defined in connection with (4). Equation (20) describes the complete time evolution, subject of course to appropriate boundary conditions, of the conditional density  $p\{\underline{x}; t | v_{t_0, t}\}$  and hence the  $\hat{h}_j(t)$ ,  $j = 1, 2, \dots, M$  from Fig. 2. The operator portion  $L^+\{\cdot\}$  represents the Fokker-Planck equation for the shot process alone while the second term in (20) is due to the effect of noisy observations on the state  $\underline{x}(t)$ . This expression is quite similar<sup>9</sup> to that which would have been obtained if the point process  $u(t)$  were WGN except for different interpretations of the operator  $L^+\{\cdot\}$ . The expression given by (20) can be used to obtain the explicit structure of the estimator  $\hat{\underline{x}}(t)$  and hence the  $\hat{h}_j(t)$ ,  $j = 1, 2, \dots, M$ .

#### IV. DERIVATION OF MODIFIED FOKKER-PLANCK EQUATION

Of the various approaches to the derivation of the modified Fokker-Planck equation given by (20), the approach taken here will follow where possible one first given by Kushner,<sup>4,5</sup> which has by now become fairly standard.<sup>9,10</sup> The derivation proceeds in two steps. First, we assume the conditional probability density  $p\{\underline{x}; t | v_{t_0, t}\}$  is given and proceed to determine the effect on this quantity of an incremental change in the conditioning or observation process  $v(t)$ . Then the effects of an incremental change in the state  $\underline{x}(t)$  are considered leading finally to the modified Fokker-Planck equation described by (20). It will prove convenient at this time to replace the conditioning upon the  $v(t)$  process by conditioning upon the process  $z(t)$ , defined by  $dz(t) = v(t) dt$ , which is possible due to the one-to-one relationship between them. In particular, we have

$$p\{\underline{x}; t | v_{t_0, t+dt}\} = p\{\underline{x}; t | z_{t_0, t}, dz(t)\} \quad (23)$$

which allows us to proceed with Step 1 of the derivation.

Step 1: It follows from application of Bayes rule that

$$\begin{aligned} p\{\underline{x}; t | z_{t_0, t}, dz(t)\} &= \frac{p\{dz(t) | z_{t_0, t}, \underline{x}(t)\} p\{\underline{x}; t | z_{t_0, t}\}}{p\{dz(t) | z_{t_0, t}\}} \\ &= \frac{p\{dz(t) | \underline{x}(t)\} p\{\underline{x}; t | z_{t_0, t}\}}{\int_{R^n} p\{dz(t) | \underline{x}(t)\} p\{\underline{x}; t | z_{t_0, t}\} d\underline{x}} \quad (24) \end{aligned}$$

where the integration is over the  $n$ -dimensional real Euclidean space  $R^n$  and we have made use of the fact that

$$p\{dz(t) | z_{t_0, t}, \underline{x}(t)\} = p\{dz(t) | \underline{x}(t)\} \quad (25)$$

since conditioning on  $z_{t_0, t}$  given  $\underline{x}(t)$  can give no new information about  $dz(t)$ .

Since

$$dz(t) = n(t) dt = [\langle \underline{c}, \underline{x}(t) \rangle + w(t)] dt \quad (26)$$

it follows that given  $\underline{x}(t)$ ,  $dz(t)$  is Gaussian with mean

$$E\{dz(t) | \underline{x}(t)\} = \langle \underline{c}, \underline{x}(t) \rangle dt \quad (27)$$

and variance

$$\text{var}\{dz(t) | \underline{x}(t)\} = \sigma_o^2 dt, \quad (28)$$

where  $\sigma_o^2$  is the variance of the WGN process  $w(t)$  as described previously. Thus,

$$p\{dz(t) | \underline{x}(t)\} = \frac{1}{\sqrt{2\pi\sigma_o^2 dt}} \exp \left\{ -\frac{1}{2\sigma_o^2 dt} [dz(t) - \langle \underline{c}, \underline{x}(t) \rangle dt]^2 \right\}. \quad (29)$$

Now defining

$$\gamma [dz(t), dt; t] = \frac{p\{\underline{x}; t | z_{t_0, t+dt}\}}{p\{\underline{x}; t | z_{t_0, t}\}}, \quad (30)$$

it follows from (24) together with (29) after canceling common factors in both the numerator and denominator that

$$\gamma [dz(t), dt; t] = \frac{\exp\{\sigma_o^{-2} dz(t) \langle \underline{c}, \underline{x}(t) \rangle - 0.5\sigma_o^{-2} \langle \underline{c}, \underline{x}(t) \rangle^2 dt\}}{\int_{R^n} \exp\{\sigma_o^{-2} dz(t) \langle \underline{c}, \underline{x} \rangle - 0.5\sigma_o^{-2} \langle \underline{c}, \underline{x} \rangle^2 dt\} p\{\underline{x}; t | z_{t_0, t}\} d\underline{x}}. \quad (31)$$

Now expanding  $\gamma [dz(t), dt; t]$  in a Taylor series about  $dz(t) = dt = 0$  it can be shown by fairly standard arguments<sup>9,10</sup> that up to terms\*  $o(dt)$

$$\gamma [dz(t), dt; t] = 1 + \frac{1}{\sigma_o^2} [dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt] \cdot [\langle \underline{c}, \underline{x}(t) \rangle - \langle \underline{c}, \hat{\underline{x}}(t) \rangle], \quad (32)$$

where  $\hat{\underline{x}}(t)$  is the conditional or minimum mean-squared estimate given by (18) or equivalently (19). Substituting this result into the defining equation (30) for  $\gamma [dz(t), dt; t]$ , we have

$$p\{\underline{x}; t | z_{t_0, t+dt}\} = p\{\underline{x}; t | z_{t_0, t}\} + dq [\underline{x}; t], \quad (33)$$

where

$$dq [\underline{x}; t] = \frac{p\{\underline{x}; t | z_{t_0, t}\}}{\sigma_o^2} [dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt] \cdot [\langle \underline{c}, \underline{x}(t) \rangle - \langle \underline{c}, \hat{\underline{x}}(t) \rangle]. \quad (34)$$

The expression given by (33) then represents the incremental change in the conditional density  $p\{\underline{x}; t | z_{t_0, t}\}$  (or equivalently  $p\{\underline{x}; t | v_{t_0, t}\}$ ) due to an incremental change in the observation alone. The derivation of the modified Fokker-Planck equation is completed in Step 2 by determining the effect upon  $p\{\underline{x}; t | z_{t_0, t}\}$  of an incremental change in the value of the state vector  $\underline{x}(t)$ .

Step 2: It is here that we depart from the Kushner derivation which depends heavily upon the higher order nature in  $dt$  of the higher order moments of  $d\underline{x}(t)$ . Such is not the case for the low-density shot processes being considered here. As an alternative, we present an approach suggested by Snyder.<sup>2</sup> In particular, observe that

$$p\{\underline{x}; t + dt | z_{t_0, t+dt}\} d\underline{x} = \Pr \left\{ \underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt} \right\}, \quad (35)$$

\*By the notation  $o(dt)$  we mean a quantity such that  $\lim_{dt \rightarrow 0} [o(dt)/dt] = 0$ .



where  $I_{\underline{x}}^n$  is a region of volume  $d\underline{x}$  centered at the point  $\underline{x}$  in  $R^n$ . Let us assume that  $dt$  is so small that the probability of more than one impulse occurring in the interval  $(t, t + dt)$  is  $o(dt)$ . Then up to higher order terms in  $dt$  we have

$$\begin{aligned} \Pr\left\{\underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt}\right\} &= [1 - \lambda(t) dt] \Pr\left\{\underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt}, dN(t) = 0\right\} \\ &+ \lambda(t) dt \Pr\left\{\underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt}, dN(t) = 1\right\}, \end{aligned} \quad (36)$$

where as before  $\lambda(t)$  is the intensity of the driving Poisson point process\* and  $dN(t)$  is the number of impulses which have occurred in the infinitesimal interval  $(t, t + dt)$ . Observe that if  $dN(t) = 0$ ,

$$\underline{x}(t + dt) = \underline{\Phi}(dt) \underline{x}(t), \quad (37)$$

where  $\underline{\Phi}(\tau) = \exp\{\underline{A}\tau\}$  is the state transition matrix associated with the linear system generating the shot process. Since  $\underline{\Phi}(\tau)$  is nonsingular<sup>11</sup>

$$\Pr\left\{\underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt}, dN(t) = 0\right\} = \Pr\left\{\underline{x}(t) \in \underline{\Phi}^{-1}(dt) I_{\underline{x}}^n | z_{t_0, t+dt}\right\}, \quad (38)$$

where by the symbolic notation  $\underline{\Phi}^{-1}(dt) I_{\underline{x}}^n$  we mean the set  $\{\underline{\xi} \in R^n; \underline{\Phi}(dt) \underline{\xi} \in I_{\underline{x}}^n\}$ . Similarly, if  $dN(t) = 1$ ,

$$\begin{aligned} \underline{x}(t + dt) &= \underline{\Phi}(dt) \underline{x}(t) + \int_t^{t+dt} \underline{\Phi}(t + dt - \sigma) \underline{b} u(\sigma) d\sigma \\ &= \underline{\Phi}(dt) \underline{x}(t) + u \underline{\Phi}(t + dt - \tau) \underline{b}, \end{aligned} \quad (39)$$

where  $\tau$  is the time of occurrence of the single impulse with  $t < \tau \leq t + dt$ , and  $u$  is a generic random variable with probability density<sup>†</sup>  $f(\cdot)$  representing the random amplitude of the exciting pulse train. Since

$$\underline{\Phi}(\tau) = \underline{I} + \tau \underline{A} + \frac{\tau^2}{2!} \underline{A}^2 + \dots \quad (40)$$

with  $\underline{I}$  the  $n \times n$  identity matrix, it follows that it is possible to write (39) as

$$\underline{x}(t + dt) = \underline{\Phi}(dt) \underline{x}(t) + \underline{b} u + O(dt) \quad (41)$$

so that finally after retaining terms up to<sup>‡</sup>  $O(dt)$

$$\begin{aligned} \Pr\left\{\underline{x}(t + dt) \in I_{\underline{x}}^n | z_{t_0, t+dt}, dN(t) = 1\right\} \\ = \int_{-\infty}^{\infty} \Pr\left\{\underline{x}(t) \in \underline{\Phi}^{-1}(dt) [\underline{I}_{\underline{x}}^n - \underline{b} u] | z_{t_0, t+dt}\right\} f(u) du, \end{aligned} \quad (42)$$

where again by  $\underline{\Phi}^{-1}(dt) [\underline{I}_{\underline{x}}^n - \underline{b} u]$  we mean the set  $\{\underline{\xi} \in R^n; \underline{\Phi}(dt) \underline{\xi} + \underline{b} u \in I_{\underline{x}}^n\}$ . Thus up to higher

\* Observe that the driving point process need not be Poisson for the development to follow.

† We will restrict attention to  $f(\cdot)$  described by either (7) or (8).

‡ By the notation  $O(dt)$  we mean a quantity such that  $\lim_{dt \rightarrow 0} [O(dt)/dt] < \infty$ .



order terms in  $dt$  we have obtained

$$\begin{aligned} p\{\underline{x}; t + dt | z_{t_0, t+dt}\} d\underline{x} = [1 - \lambda(t) dt] \Pr\left\{\underline{x}(t) \in \underline{\Phi}^{-1}(dt) I_{\underline{x}}^n | z_{t_0, t+dt}\right\} \\ + \lambda(t) dt \int_{-\infty}^{\infty} \Pr\left\{\underline{x}(t) \in \underline{\Phi}^{-1}(dt) [I_{\underline{x}}^n - \underline{b}u] | z_{t_0, t+dt}\right\} f(u) du \end{aligned} \quad (43)$$

from which it follows that

$$\begin{aligned} p\{\underline{x}; t + dt | z_{t_0, t+dt}\} = [1 - \lambda(t) dt] \det^{-1} \underline{\Phi}(dt) p\{\underline{\Phi}^{-1}(dt) \underline{x}; t | z_{t_0, t+dt}\} \\ + \lambda(t) dt \det^{-1} \underline{\Phi}(dt) \int_{-\infty}^{\infty} p\{\underline{\Phi}^{-1}(dt) [\underline{x} - \underline{b}u]; t | z_{t_0, t+dt}\} f(u) du \end{aligned} \quad (44)$$

where  $\det \underline{\Phi}(dt)$  represents the Jacobian of the transformation relating  $\underline{x}(t + dt)$  to  $\underline{x}(t)$ . At this point some approximations are in order. By expanding the first conditional probability on the right-hand side of (44) in a Taylor series about the point  $\underline{x}$ , we have

$$p\{\underline{\Phi}^{-1}(dt) \underline{x}; t | z_{t_0, t+dt}\} = p\{\underline{x}; t | z_{t_0, t+dt}\} - \{\underline{A} \underline{x}\}^T \frac{\partial p\{\underline{x}; t | z_{t_0, t+dt}\}}{\partial \underline{x}} dt + o(dt) \quad (45)$$

where the superscript  $T$  denotes transpose. Similarly, we have

$$p\{\underline{\Phi}^{-1}(dt) [\underline{x} - \underline{b}u]; t | z_{t_0, t+dt}\} = p\{\underline{x} - \underline{b}u; t | z_{t_0, t+dt}\} + O(dt) \quad (46)$$

where  $O(dt)$  represents a quantity such that  $O(dt)/dt \rightarrow M$  a nonzero constant as  $dt \rightarrow 0$ . Finally, after observing that

$$\det^{-1} \underline{\Phi}(dt) = 1 - \text{Tr}\{\underline{A}\} dt + o(dt) \quad (47)$$

it follows that upon substituting these results into (44) and grouping together terms in  $dt$

$$\begin{aligned} p\{\underline{x}; t + dt | z_{t_0, t+dt}\} = \{1 - [\lambda(t) + \text{Tr} \underline{A}] dt\} p\{\underline{x}; t | z_{t_0, t+dt}\} \\ - \{\underline{A} \underline{x}\}^T \frac{\partial}{\partial \underline{x}} p\{\underline{x}; t | z_{t_0, t+dt}\} dt \\ + \lambda(t) dt \int_{-\infty}^{\infty} p\{\underline{x} - \underline{b}u; t | z_{t_0, t+dt}\} f(u) du + o(dt) \end{aligned} \quad (48)$$

which is more compactly written as

$$\begin{aligned} p\{\underline{x}; t + dt | z_{t_0, t+dt}\} = p\{\underline{x}; t | z_{t_0, t+dt}\} - \text{Tr} \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t+dt}\} \right) dt \\ - \lambda(t) dt \int_{-\infty}^{\infty} [p\{\underline{x}; t | z_{t_0, t+dt}\} - p\{\underline{x} - \underline{b}u; t | z_{t_0, t+dt}\}] f(u) du + o(dt) \end{aligned} \quad (49)$$

Finally, defining

$$dp\{\underline{x}; t | z_{t_0, t}\} \triangleq p\{\underline{x}; t + dt | z_{t_0, t+dt}\} - p\{\underline{x}; t | z_{t_0, t}\} \quad (50)$$

we have upon neglecting higher order terms in  $dt$

$$\begin{aligned} dp\{\underline{x}; t | z_{t_0, t}\} &= dq[\underline{x}; t] - \text{Tr} \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t+dt}\} \right) dt \\ &- \lambda(t) dt \int_{-\infty}^{\infty} \left[ p\{\underline{x}; t | z_{t_0, t+dt}\} - p\{\underline{x} - \underline{b}u; t | z_{t_0, t+dt}\} \right] f(u) du \end{aligned} \quad (51)$$

At this point it is possible to divide both sides of this last expression by  $dt$  and pass to the limit as  $dt \rightarrow 0$  with the result

$$\begin{aligned} \frac{\partial p\{\underline{x}; t | v_{t_0, t}\}}{\partial t} &= L^+ p\{\underline{x}; t | v_{t_0, t}\} + \sigma_o^{-2} p\{\underline{x}; t | v_{t_0, t}\} \\ &\times [v(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle] [\langle \underline{c}, \underline{x}(t) \rangle - \langle \underline{c}, \hat{\underline{x}}(t) \rangle] \end{aligned} \quad (52)$$

which is the desired result [Eq. (20)]. The operator  $L^+$  which represents the forward differential generator of the shot process has been defined previously in (21) and the sequel.

## V. LEAST-SQUARE ESTIMATOR EQUATIONS

Differential equations for the least-squares estimator  $\hat{\underline{x}}(t)$  given by (19) can be readily obtained from the results of the preceding section. In particular, if we multiply both sides of the modified Fokker-Planck equation (51) by  $\underline{x}$  and integrate over  $R^n$  we obtain, as shown in Appendix A,

$$\dot{\hat{\underline{x}}}(t) = \underline{A} \hat{\underline{x}}(t) + m_u \underline{b} \lambda(t) + \sigma_o^{-2} \underline{K}(t) \underline{c} \{v(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle\} \quad (53)$$

Here the gain matrix  $\underline{K}(t)$  is given by the variance of the estimation error

$$\underline{K}(t) = E \left\{ [\underline{x}(t) - \hat{\underline{x}}(t)] [\underline{x}(t) - \hat{\underline{x}}(t)]^T | v_{t_0, t} \right\} \quad (54)$$

and  $m_u = E\{u\}$  is the mean of the pulse amplitudes — actually zero for the distributions described by Eqs. (7) and (8). Figure 3 illustrates the implementation of the estimator  $\hat{\underline{x}}(t)$  under the

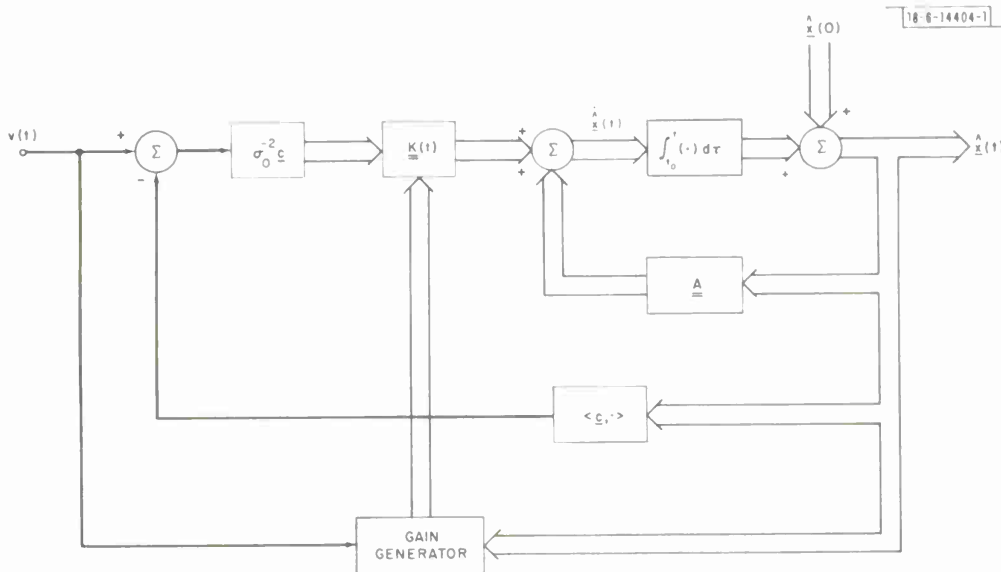


Fig. 3. Minimum mean-square estimator.

assumption  $m_u = 0$ . The structure is exactly that of the Kalman-Bucy filter operating on Gauss-Markov data, the only difference being in the way the gain matrix  $\underline{K}(t)$ , or equivalently the error covariance matrix, is obtained. As in the Kalman-Bucy theory, a differential equation for the error covariance matrix can be obtained from the results of the preceding section. In Appendix A it is shown that the error covariance matrix  $\underline{K}(t)$  satisfies the matrix differential equation

$$\begin{aligned} \underline{K}(t) = & \underline{A} \underline{K}(t) + \underline{K}(t) \underline{A}^T - \lambda(t) [\sigma_u^2 + m_u^2] \underline{B} - \sigma_o^{-2} \underline{K}(t) \underline{C} \underline{K}(t) \\ & + \sigma_o^{-2} [v(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle] E \left\{ [\underline{x}(t) - \hat{\underline{x}}(t)] [\underline{x}(t) - \hat{\underline{x}}(t)]^T \right. \\ & \left. \times [\underline{x}(t) - \hat{\underline{x}}(t)]^T \underline{c} | v_{t_0, t} \right\} \end{aligned} \quad (55)$$

Here  $\underline{B} = \underline{b} \underline{b}^T$ ,  $\underline{C} = \underline{c} \underline{c}^T$ , and  $\sigma_u^2$  is the variance of the pulse amplitudes which excite the linear dynamical system to obtain the shot process. Observe that if the Poisson point process  $u(t)$  were replaced by a WGN process the last term in (55) vanishes since odd moments (in this case the third) of Gaussian zero-mean variates are identically zero. In the case we have treated, these odd moments need not vanish and the gain matrix  $\underline{K}(t)$  depends explicitly upon both the input data and the current estimate  $\hat{\underline{x}}(t)$  through this last term. This results in a nonlinear receiver structure and considerably complicates the receiver implementation since now the gain cannot be determined a priori before observations are taken. Indeed, determination of  $\underline{K}(t)$  from (55) requires evaluation of third moments of the estimation error. It can be shown that differential equations can be obtained for the third and higher order moments which will be found to depend upon still higher order moments of the estimation error as well as the noisy observations. Snyder<sup>2</sup> has suggested that truncation of this dependency on higher order moments at some point will result in a reasonable suboptimum detector structure. Such questions are objects of continuing investigations.

## VI. SUMMARY AND CONCLUSIONS

We have explicitly defined the optimum detector structure for reception of M-ary orthogonal signals in ELF noise environments. This structure has been shown to bear some resemblance to that which would have been obtained in the presence of Gaussian noise alone. The presence of the low-density shot component has been shown to severely complicate the receiver implementation. Studies in progress will be concerned with the implementation and performance of several suboptimum versions of this structure obtained as described in the preceding section by truncating the dependency of the gain matrix on higher order moments at some level. Also under study are techniques for obtaining exponential error bounds on error probability performance over the ELF channel and adaptive identification techniques for determining the linear system dynamics and the time-varying intensity of the Poisson point process from observations. Another area which has been under continuing study is that of the evaluation of the relative performance of various suboptimum nonlinear detector structures of the limiting variety. These results will be described in a later report.



# APPENDIX A

## DERIVATION OF LEAST-SQUARES ESTIMATOR EQUATIONS

Here we desire to develop a differential equation for the least-squares estimator  $\hat{\underline{x}}(t)$  given by (19). First define the quantity

$$\begin{aligned} d\hat{\underline{x}}(t) &\triangleq \hat{\underline{x}}(t + dt) - \hat{\underline{x}}(t) \\ &= \int_{R^n} \underline{x} dp\{\underline{x}; t | z_{t_0, t}\} d\underline{x} \end{aligned} \quad (A-1)$$

so that from (51)

$$\begin{aligned} d\hat{\underline{x}}(t) &= \int_{R^n} \underline{x} dq[\underline{x}; t] d\underline{x} - dt \int_{R^n} \underline{x} \text{Tr} \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t}\} \right) d\underline{x} \\ &\quad - \lambda(t) dt \int_{R^n} \underline{x} \left\{ \int_{-\infty}^{\infty} [p\{\underline{x}; t | z_{t_0, t}\} - p\{\underline{x} - \underline{b}u; t | z_{t_0, t}\}] f(u) du \right\} d\underline{x} \quad . \end{aligned} \quad (A-2)$$

Now evaluating each of these integrals separately it follows first from (34) that

$$\int_{R^n} \underline{x} dq[\underline{x}; t] d\underline{x} = \sigma_o^2 \underline{K}(t) \underline{c}\{dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt\} \quad , \quad (A-3)$$

where  $\underline{K}(t)$  is the covariance of the estimation error given by

$$\begin{aligned} \underline{K}(t) &= \int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T p\{\underline{x}; t | z_{t_0, t}\} d\underline{x} \\ &= E\{[\underline{x}(t) - \hat{\underline{x}}(t)] [\underline{x}(t) - \hat{\underline{x}}(t)]^T | z_{t_0, t}\} \quad . \end{aligned} \quad (A-4)$$

Similarly, if we let  $\{\underline{A} \underline{x}\}_i$  denote the  $i^{\text{th}}$  component  $i = 1, 2, \dots, n$  of  $\underline{A} \underline{x}$ , then under the assumption

$$x_i \{\underline{A} \underline{x}\}_i p\{\underline{x}; t | z_{t_0, t}\} = 0 \quad ; \quad i = 1, 2, \dots, n \quad , \quad (A-5)$$

whenever  $x_i = \pm \infty$ , it follows after integration by parts that

$$\int_{R^n} \underline{x} \text{Tr} \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t}\} \right) d\underline{x} = -\underline{A} \hat{\underline{x}}(t) \quad . \quad (A-6)$$

Finally, after some algebra it can be shown that the last integral in (A-2) can be expressed as

$$\int_{R^n} \underline{x} \left\{ \int_{-\infty}^{\infty} [p\{\underline{x}; t | z_{t_0, t}\} - p\{\underline{x} - \underline{b}u; t | z_{t_0, t}\}] f(u) du \right\} d\underline{x} = -\underline{b} m_u \quad , \quad (A-7)$$

where  $m_u = E\{u\}$  is the mean value of the exciting pulse amplitudes. Thus, the expression (A-1) for  $d\hat{\underline{x}}(t)$  becomes

$$d\hat{\underline{x}}(t) = \underline{A} \hat{\underline{x}}(t) dt + m_u \underline{b} \lambda(t) dt + \sigma_o^{-2} \underline{K}(t) \underline{c}\{dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt\} \quad (A-8)$$

so that, after dividing both sides by  $dt$  and making use of the fact that  $dz(t) = v(t) dt$ , we arrive at Eq. (53) of the text.

## APPENDIX B

### ERROR COVARIANCE EQUATION

Let us first observe the vector-matrix identity

$$\begin{aligned} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T &= [\underline{x} - \hat{\underline{x}}(t + dt)] [\underline{x} - \hat{\underline{x}}(t + dt)]^T + d\hat{\underline{x}}(t) d\hat{\underline{x}}^T(t) \\ &\quad + [\underline{x} - \hat{\underline{x}}(t + dt)] d\hat{\underline{x}}^T(t) + d\hat{\underline{x}}(t) [\underline{x} - \hat{\underline{x}}(t + dt)]^T, \end{aligned} \quad (B-1)$$

where  $d\hat{\underline{x}}(t) = \hat{\underline{x}}(t + dt) - \hat{\underline{x}}(t)$ . Now multiplying the right-hand side of this last expression by the left-hand side of (51), and the left-hand side by the right-hand side of (51), we obtain (see Ref. 9 for similar results), after integrating over  $R^n$ ,

$$\begin{aligned} d\underline{K}(t) + d\hat{\underline{x}}(t) d\hat{\underline{x}}^T(t) &= -dt \int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T \text{Tr} \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t+dt}\} \right) d\underline{x} \\ &\quad - \lambda(t) dt \int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T E \left[ \Delta_u p\{\underline{x}; t | z_{t_0, t+dt}\} \right] d\underline{x} \\ &\quad + \sigma_o^{-2} [dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt] \int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T \langle \underline{c}, \underline{x}(t) \rangle \\ &\quad - \langle \underline{c}, \hat{\underline{x}}(t) \rangle p\{\underline{x}; t | z_{t_0, t}\} d\underline{x}, \end{aligned} \quad (B-2)$$

where

$$\begin{aligned} \underline{K}(t) &= \int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T p\{\underline{x}; t | z_{t_0, t}\} d\underline{x} \\ &= E\{[\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T | z_{t_0, t}\}, \end{aligned} \quad (B-3)$$

and  $d\underline{K}(t) = \underline{K}(t + dt) - \underline{K}(t)$ . Evaluating the first integral on the right-hand side of (B-2) by parts, again under similar assumptions as stated in (A-5), it follows that

$$\int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T \left( \frac{\partial}{\partial \underline{x}} \underline{A} \underline{x} p\{\underline{x}; t | z_{t_0, t}\} \right) d\underline{x} = -[\underline{A} \underline{K}(t) + \underline{K}(t) \underline{A}^T] \quad (B-4)$$

Similarly, it can be shown easily that

$$\int_{R^n} [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T E \left[ \Delta_u p\{\underline{x}; t | z_{t_0, t+dt}\} \right] d\underline{x} = [\sigma_u^2 + m_u^2] \underline{B} \quad (B-5)$$

where  $\sigma_u^2 = \text{var}\{u\}$  is the variance of the pulse amplitudes represented by the generic random variable  $u$  and  $\underline{B} = \underline{b} \underline{b}^T$ . Thus it follows

$$\begin{aligned} d\underline{K}(t) + d\hat{\underline{x}}(t) d\hat{\underline{x}}^T(t) &= [\underline{A} \underline{K}(t) + \underline{K}(t) \underline{A}^T] dt - \lambda(t) dt [\sigma_u^2 + m_u^2] \underline{B} \\ &\quad + \sigma_o^{-2} [dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt] E \left\{ [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T [\underline{x} - \hat{\underline{x}}(t)]^T \underline{c} | z_{t_0, t} \right\}. \end{aligned} \quad (B-6)$$

Observe from (A-8) that after dropping obvious terms of higher order in  $dt$

$$d\hat{\underline{x}}(t) d\hat{\underline{x}}^T(t) = \sigma_o^{-4} [dz(t) - \langle \underline{c}, \hat{\underline{x}} \rangle dt]^2 \underline{K}(t) \underline{C} \underline{K}(t) \quad (B-7)$$

where  $\underline{\underline{C}} = \underline{c} \underline{c}^T$ . Furthermore, since

$$dz(t) = \langle \underline{c}, \underline{x}(t) \rangle dt + d\xi(t) \quad , \quad (B-8)$$

where  $d\xi(t) = w(t) dt$  and  $\xi(t)$  is a Brownian motion process, we have up to terms  $o(dt)$

$$\begin{aligned} d\hat{\underline{x}}(t) d\hat{\underline{x}}^T(t) &= \sigma_o^{-4} d^2\xi(t) \underline{\underline{K}}(t) \underline{\underline{C}} \underline{\underline{K}}(t) \\ &= \sigma_o^{-2} dt \underline{\underline{K}}(t) \underline{\underline{C}} \underline{\underline{K}}(t) \quad , \end{aligned} \quad (B-9)$$

where we have made use of the fact<sup>9</sup> that the process  $d^2\xi(t)$  is essentially deterministic and equal to  $\sigma_o^2 dt$ .

Finally, we obtain

$$\begin{aligned} d\underline{\underline{K}}(t) &= [\underline{\underline{A}} \underline{\underline{K}}(t) + \underline{\underline{K}}(t) \underline{\underline{A}}^T] dt - \lambda(t) dt [\sigma_u^2 + m_u^2] \underline{\underline{B}} - \sigma_o^{-2} dt \underline{\underline{K}}(t) \underline{\underline{C}} \underline{\underline{K}}(t) \\ &+ \sigma_o^{-2} [dz(t) - \langle \underline{c}, \hat{\underline{x}}(t) \rangle dt] E \left\{ [\underline{x} - \hat{\underline{x}}(t)] [\underline{x} - \hat{\underline{x}}(t)]^T [\underline{x} - \hat{\underline{x}}(t)]^T \underline{c} | z_{t_0, t} \right\} dt \quad . \end{aligned} \quad (B-10)$$

whereupon, dividing both sides of this last expression by  $dt$  while passing to the limit as  $dt \rightarrow 0$  and making use of the fact  $dz(t) = v(t) dt$ , we obtain Eq. (55) of the text.

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13. ABSTRACT  <p>Explicit receiver structures for the optimum detection of M-ary orthogonal signals in impulsive noise environments typical of ELF are determined. The resulting optimum structures, while shown to bear some resemblance to that which would have been obtained in the presence of Gaussian noise alone, exhibit an interesting nonlinear behavior. The difficulties in the actual implementation of this receiver are discussed, and several definitely suboptimum yet computationally superior structures are suggested. The results are felt to be applicable to a much broader class of channels characterized by impulsive interference.</p>										
14. KEY WORDS										
<table><tbody><tr><td>ELF noise</td><td>M-ary detector structures</td></tr><tr><td>non-Gaussian noise</td><td>receiver structures</td></tr><tr><td>impulsive noise</td><td>least-square estimators</td></tr><tr><td>shot noise</td><td></td></tr></tbody></table>			ELF noise	M-ary detector structures	non-Gaussian noise	receiver structures	impulsive noise	least-square estimators	shot noise	
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